SOME $\mathbb{Z}_3^n$-EQUIVARIANT TRIANGULATIONS OF $\mathbb{CP}^n$

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Abstract. In 1983, Banchoff and Kuhnel constructed a minimal triangulation of $\mathbb{CP}^2$ with 9 vertices. $\mathbb{CP}^3$ was first triangulated by Bagchi and Datta in 2012 with 18 vertices. Known lower bound on number of vertices of a triangulation of $\mathbb{CP}^n$ is $1+\frac{(n+1)^2}{2}$ for $n \geq 3$. We give explicit construction of some triangulations of complex projective space $\mathbb{CP}^n$ with $\frac{4^{n+1}-1}{3}$ vertices for all n. No explicit triangulation of $\mathbb{CP}^n$ is known for $n \geq 4$.

1. Introduction

The space of complex lines through the origin of the $(n+1)$-dimensional complex vector space $\mathbb{C}^{n+1}$ is called complex projective space, denoted by $\mathbb{CP}^n$, of dimension n. These can be defined in the following way too. Consider the $(2n+1)$-dimensional sphere $S^{2n+1} = \{(z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1} : |z_0|^2 + \cdots + |z_n|^2 = 1\}$. The circle $\mathbb{T}^1 = S^1$ acts on S^{2n+1} by

$$t \cdot (z_0, z_1, \ldots, z_n) \to (tz_0, tz_1, \ldots, tz_n).$$

The quotient space $S^{2n+1}/\mathbb{T}^1$ can be identified with $\mathbb{CP}^n$. We briefly recall the topological construction of $\mathbb{CP}^n$ in subsection 2.2. $\mathbb{T}^n$ acts naturally on $\mathbb{CP}^n$. This induces a natural $\mathbb{Z}_3^n \subset \mathbb{T}^n$ action on $\mathbb{CP}^n$.

There are lower bounds on the number of vertices of a triangulation of $\mathbb{CP}^n$ due to Arnoux and Marin [AM91]. Starting with the unique minimum triangulation of $\mathbb{T}^2$, Banchoff and Kuhnel constructed a 10-vertex triangulation of $\mathbb{CP}^2$ [BK92]. Considering $\mathbb{CP}^3$ as the orbit space of the natural S_3-action on $S^2 \times S^2 \times S^2$, Bagchi and Datta constructed 30 vertex triangulation of $\mathbb{CP}^3$, [BD12]. Any explicit triangulation for $\mathbb{CP}^n$ is not known for $n \geq 4$. Here we construct some natural $\mathbb{Z}_3$-equivariant triangulation of complex projective space $\mathbb{CP}^n$ and some toric manifolds.

2. Preliminaries

2.1. **Cubical subdivision of simple polytopes.** Codimension one faces of a convex polytope are called *facets*. An n-dimensional *simple polytope* in $\mathbb{R}^n$ is a convex polytope where exactly n facets meet at each vertex. The ready examples of simple polytopes are simplices and cubes.

We discuss the *cubical subdivision* of an n-dimensional simple polytope Q. For a face σ Q, let C_σ be the center point of σ. Let I_σ be the convex hull of

$$\{C_\tau : \tau \text{ is a face of } Q \text{ containing } \sigma\}.$$

If codimension of σ is zero, i.e., $\sigma = Q$, then $I_\sigma = C_Q$. Let σ be a face of codimension $k > 0$. Since Q is a simple polytope, $\sigma = F_{i_1} \cap \cdots \cap F_{i_k}$ for a unique collection of k many facets $\{F_{i_1}, \ldots, F_{i_k}\}$ of Q. Note that $C_Q \in I_\sigma$. Also if $F = G_{j_1} \cap \cdots \cap G_{j_l}$ with $\{G_{j_1}, \ldots, G_{j_l}\} \subseteq \{F_{i_1}, \ldots, F_{i_k}\}$ then $C_F \in I_\sigma$. So number of vertices of I_σ is 2^{n-k}. That

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is I_σ is an $(n-k)$-dimensional cube in Q. If σ is a face of τ then I_τ is a face of I_σ. Let $\mathcal{L}(Q)$ be the set of faces of Q. Then

$$Q = \bigcup_{\sigma \in \mathcal{L}(Q)} I_\sigma. \tag{2.1}$$

This gives a cubical subdivision of Q. A cubical subdivision of triangle and tetrahedron are shown in the Figure 1.

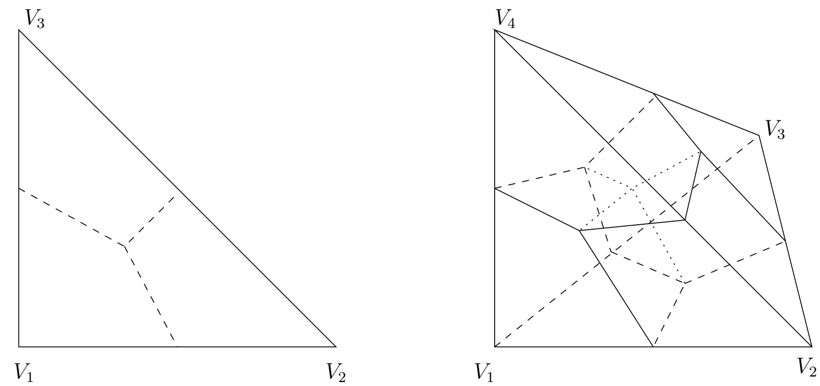

FIGURE 1.

2.2. **Construction of some toric manifolds.** Following [DJ91], we construct some toric manifolds. Let $\mathcal{F}(Q) = \{F_1, \ldots, F_m\}$ be the facets of an n-dimensional simple polytope Q. Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{Z}^n$ and $e_0 = e_1 + \cdots + e_n$.

Definition 2.1. A function $\xi : \mathcal{F}(Q) \to \{e_0, \ldots, e_n\}$ is called standard characteristic function of Q if $\xi(F_{i_1}), \ldots, \xi(F_{i_n})$ are distinct whenever $F_{i_1} \cap \cdots \cap F_{i_n}$ is nonempty.

From the standard characteristic function ξ of the simple polytope Q, we construct a toric manifold. Let σ be a codimension-k face of Q. So $F = F_{i_1} \cap \cdots \cap F_{i_k}$ for a unique collection of facets $F_{i_1}, \ldots, F_{i_k}$. Let $\mathbb{Z}_\sigma$ be the submodule of $\mathbb{Z}^n$ generated by $\{\xi(F_{i_1}), \ldots, \xi(F_{i_k})\}$. So $\mathbb{Z}_\sigma$ is a direct summand of $\mathbb{Z}^n$. Let $\mathbb{T}_\sigma$ be the subtorus of $\mathbb{T}^n$ determined by $\mathbb{Z}_\sigma$. We define an equivalence relation $\sim$ on $\mathbb{T}^n \times Q$ by

$$(t, p) \sim (s, q) \text{ if and only if } p = q \text{ and } s^{-1}t \in \mathbb{T}_\sigma, \tag{2.2}$$

where σ is the smallest face containing p in its relative interior. Then the quotient space $M(Q, \xi) := (\mathbb{T}^n \times Q)/\sim$ is a toric manifold, see [DJ91] for details. We denote the equivalence class of (t, p) by $[(t, p)]^\sim$. The group operation on $\mathbb{T}^n$ induces an action of $\mathbb{T}^n$ on $M(Q, \xi)$. The orbit map

$$\pi : M(Q, \xi) \to Q \text{ is defined by } \pi([(t, p)]^\sim) = p. \tag{2.3}$$

So by Equation 2.1 we get

$$M(Q, \xi) = \bigcup_{\sigma \in \mathcal{L}(Q)} \pi^{-1}(I_\sigma). \tag{2.4}$$

Remark 2.2. *Let Q be an n-simplex $\triangle^n$ with vertices $\{v_0, \ldots, v_n\}$ and F_i be the facet not containing the vertex v_i of $\triangle^n$. The function $\xi : \{F_0, \ldots, F_n\} \to \{e_0, \ldots, e_n\}$ defined by*

$$\xi(F_i) = e_i$$

is the standard characteristic function of $\triangle^n$. The toric manifold $M(\triangle^n, \xi)$ is equivariantly homeomorphic to $\mathbb{CP}^n$, see Proposition 1.8 and Example 1.18 in [DJ91].

2.3. **Triangulations on manifolds.** We recall some basic definitions for triangulation of manifolds. A compact convex polyhedron in some R^m which spans an affine subspace of dimension n is called an n-cell. So we can define faces of an n-cell.

Definition 2.3. A *cell complex* X is a finite collection of cells in some $\mathbb{R}^n$ satisfying, (i) if B is a face of A and $A \in X$ then $B \in X$, (ii) if $A, B \in X$ and $A \cap B$ is nonempty then $A \cap B$ is a face of both A and B. Zero dimensional cells of X are called vertices of X. A cell complex X is *simplicial* if each $A \in X$ is a simplex.

We may denote a cell σ with vertices $\{v_1, \dots, v_k\}$ by $v_1 v_2 \dots v_k$. The vertex set of a simplicial complex X is denoted by $V(X)$.

Definition 2.4. Let X, Y are two simplicial complexes. Then a *simplicial embedding* from X to Y is an injection $\eta : X \to Y$ such that $\eta(\sigma)$ is a face of Y if σ is a face of X.

Definition 2.5. Two simpicial complex X and Y are said to be isomorphic if there is a bijection $\eta : X \to Y$ such that $\eta(\sigma)$ is a face of Y if and only if σ is a face of X.

The union of all cells in a simplicial complex X is called the *geometric carrier* of X which is denoted by $|X|$.

Definition 2.6. If a Hausdorff topological space M is homeomorphic to $|X|$, the geometric carrier of a simplicial complex, then we say that X is a *triangulation* of M. If $|X|$ is a topological d-ball (respectively, d-sphere) then X is called a *triangulated d-ball* (resp., *triangulated d-sphere*).

For notational purpose, we may denote the triangulation of a space by the same. From the following proposition we can construct a simplicial complex from a cell complex.

Proposition 2.7 (Proposition 2.9, [RS82])**.** *A cell complex can be subdivided to a simplicial complex without introducing any new vertices.*

Definition 2.8. Let G be a finite group. A *G-equivariant triangulation* of the G-space N is a triangulation X of N such that if $\sigma \in X$, then $g(\sigma) \in X$ for all $g \in G$, where g is regarded as the homeomorphism corresponding to the action of g on $N = |X|$.

Definition 2.9. Let M be the union of cells in a cell complex X. Let a finite group G acts on M. Let D be a cell in M such that $M = \cup_{g \in G} gD$ and $gD \cap hD$ is a face of both. Then D is called a $fundamental\ cell$ in X.

3. Some $\mathbb{Z}_3^n$-equivariant triangulation of n-torus

Let $I^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : 0 \leq x_i \leq 1 \text{ for } i = 1, \dots, n\}$. Define an equivalence relation $\sim$ on I^n by

$$(x_1, \dots, x_n) \sim (y_1, \dots, y_n) \text{ if and only if } |x_i - y_i| = 0 \text{ or } 1 \text{ for } i = 1, \dots, n. \tag{3.1}$$

Then $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$ is equal to the quotient space $I^n/\sim$.

Let $\{e_1, \dots, e_n\}$ be the standard basis of $\mathbb{Z}^n \subset \mathbb{R}^n$ and $e_0 = e_1 + \cdots + e_n$. Let S^1_i be the circle subgroup of $\mathbb{T}^n$ determined by the vector $e_i \in \mathbb{Z}^n$. So the automorphism of $\mathbb{Z}^n$ defined by

$$e_i \to e_0 \text{ and } e_j \to e_j \text{ for } j \in \{1, \dots, \hat{i}, \dots, n\}$$

induces the following isomorphism, where $\hat{}$ represents the omition of the corresponding entry.

$$\mathbb{T}^n = S^1_1 \times \dots \times S^1_n \cong S^1_1 \times \dots \times S^1_{i-1} \times S^1_0 \times S^1_{i+1} \times \dots \times S^1_n =: \mathbb{T}^n_i, \tag{3.2}$$

for $i = 1, \dots, n$.

Let $\mathbb{Z}_3 = \{1, \omega, \omega^2\}$, where ω is a primitive root of $x^3 = 1$. Then $\mathbb{Z}_3$ acts freely on S^1_i by 120-degree rotation for $i \in \{0, \dots, n\}$. So $\mathbb{Z}_3^n$ acts on $\mathbb{T}^n$ by diagonal action. This induces a $\mathbb{Z}_3^n$-action on $\mathbb{CP}^n$.

Definition 3.1. The map $\mathfrak{p}_i : \mathbb{R}^n \to \mathbb{R}$ defined by $\mathfrak{p}_i(x_1, \dots, x_n) = x_i$ is called i-th coordinate function for $i = 1, \dots, n$.

Now we construct some $\mathbb{Z}_3^n$-equivariant triangulation of $\mathbb{T}^n$ with 3^n vertices. Let $a = 1/3$ and $b = 2/3$. The triangulation of I^1 as in Figure 2 gives a $\mathbb{Z}_3$-equivariant triangulation of $\mathbb{T}^1 = S^1$ with 3 vertices. Explictly the action is given by $\omega 0 = a$, $\omega(a) = b$ and $\omega(b) = 0$.

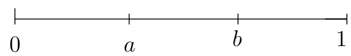

FIGURE 2.

Consider the triangulation of I^2 as in Figure 3. This induces a $\mathbb{Z}_3^2$-equivariant triangulation of $\mathbb{T}^2$ with 3^2 vertices.

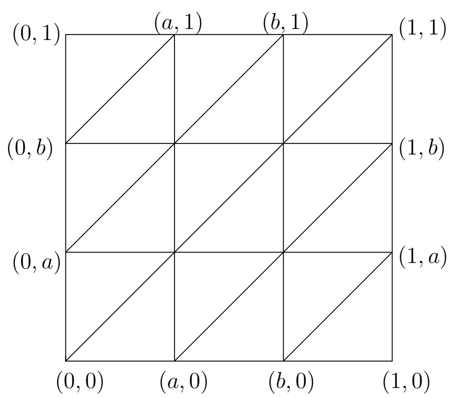

FIGURE 3.

Suppose proceeding in this way we construct a triangulation of I^{n-1} with 4^{n-1} vertices such that this induces a $\mathbb{Z}_3^{n-1}$-equivariant triangulation of $\mathbb{T}^{n-1}$ with 3^{n-1} vertices. We give inductive arguments to construct a triangulation of I^n. Let

$$I_i^{n-1} = \{(x_1, \dots, x_n) \in I^n : x_i = 0\} \text{ for } i = 1, \dots, n. \tag{3.3}$$

Consider the linear automorphism A_i of $\mathbb{R}^n$ defined by

$$A_i(x_1, \dots, x_n) = (x_1, \dots, x_{i-1}, x_n, x_{i+1}, \dots, x_{n-1}, x_i) \text{ for } i = 1, \dots, n-1. \tag{3.4}$$

Then $A_i : I_n^{n-1} \to I_i^{n-1}$ is a diffeomorphism as manifold with corners for $i = 1, \dots, n-1$. We consider the triangulation $\mathcal{I}_n^{n-1}$ of I_n^{n-1} is same as the triangulation of I^{n-1}. So A_i induces a triangulation $\mathcal{I}_i^{n-1}$ on I_i^{n-1} for $i = 1, \dots, n-1$.

Let $a_i = (0, \dots, 0, a, 0, \dots, 0)$ and $b_i = (0, \dots, 0, b, 0, \dots, 0)$ where a and b are i-th component. By translating the triangulation of I_i^{n-1}, we get a triangulation of $I_i^{n+1} + a_i$, $I_i^{n-1} + b_i$ and $I_i^{n-1} + e_i$ for $i = 1, \dots, n$. By induction, the coordinate functions are increasing along any edge of I_i^{n-1}, $I_i^{n-1} + a_i$, $I_i^{n-1} + b_i$ and $I_i^{n-1} + e_i$.

Let $I_a^n = \{(x_1, \dots, x_n) \in I^n : x_i \leq a\}$. The triangulation of I_i^{n-1} and $I_i^{n-1} + a_i$ for $i = 1, \dots, n$ induces a triangulation on the boundary of I_a^n. Since the coordinate functions along any edge of this boundary triangulation are increasing, by adding the diagonal

edge joining the vertices $(0,\dots,0)$ and $(a,\dots,a)$ of I_a^n we get a triangulation of I_a^n. Let $\mathcal{A} = \{(x_1,\dots,x_n) \in I^n : x_i \in \{0,a,b\}\}$. So we get

$$I^n = \bigcup_{v\in\mathcal{A}} I_a^n + v.$$

Then the triangulation of I_a^n induces a triangulation $\mathcal{I}^n$ of I^n with 4^n vertices. Clearly this triangulation of I^n induces a $\mathbb{Z}_3^n$-equivariant triangulation of $\mathbb{T}^n$ with 3^n vertices. Hence we get the following.

Lemma 3.2. *There is a $\mathbb{Z}_3^n$-equivariant triangulation $\mathcal{T}^n$ of $\mathbb{T}^n$ with 3^n vertices where $\mathbb{Z}_3^n$ acts on $\mathbb{T}^n$ diagonally.*

For $\{i_1,\dots,i_k\} \subset \{1,\dots,n\}$ with $i_1 < \cdots < i_k$, let

$$I(n;i_1,\dots,i_k) = \{(x_1,\dots,x_n) \in I^n : x_i = 0 \text{ if } i \notin \{i_1,\dots,i_k\}\}.$$

Let

$$\mathfrak{i} : I(n;i_1,\dots,i_k) \to I^n \tag{3.5}$$

be the inclusion and

$$\mathfrak{p} : I^n \to I(n,i_1,\dots,i_k) \tag{3.6}$$

be the projection. These maps induces the following inclusion and projection which are denoted by same respectively.

$$\mathfrak{i} : S^1_{i_1} \times \cdots \times S^1_{i_k} \to \mathbb{T}^n \tag{3.7}$$

and

$$\mathfrak{p} : \mathbb{T}^n \to S^1_{i_1} \times \cdots \times S^1_{i_k}. \tag{3.8}$$

Note that the triangulation $\mathcal{I}^n$ of I^n gives a triangulation $\mathcal{I}(n;i_1,\dots,i_k)$ on $I(n;i_1,\dots,i_k)$ with 4^k vertices. The triangulation $\mathcal{I}(n;i_1,\dots,i_k)$ induces a triangulation on $S^1_{i_1} \times \cdots \times S^1_{i_k}$. We get the following lemma.

Lemma 3.3. *With the above triangulation of $S^1_{i_1} \times \cdots \times S^1_{i_k}$ and $\mathbb{T}^n$, the maps $\mathfrak{i}$ and $\mathfrak{p}$ are simplicial.*

Now we want to construct some triangulations of $\mathbb{T}_i^n$ such that the simplices are same as $\mathbb{T}^n$ and the homeomorphism in Equation 3.2 is $\mathbb{Z}_3^n$-equivariant with these triangulation of $\mathbb{T}_i^n$ and $\mathbb{T}^n$.

First we discuss another description of $\mathbb{T}_i^n$ for $i = 1,\dots,n$. Let

$$J_i^n = \{\epsilon_1 e_1 + \cdots + \epsilon_{i-1} e_{i-1} + \epsilon_i e_0 + \epsilon_{i+1} e_{i+1} + \cdots + \epsilon_n e_n : \epsilon_i \in \{0,1\} \text{ for } i = 1,\dots,n\}.$$

Let I_i^n be the convex hull of J_i^n in $\mathbb{R}^n$. Define an equivalence relation $\sim_i$ on I_i^n by

$$(x_1,\dots,x_n) \sim_i (y_1,\dots,y_n) \text{ if and only if } |x_i - y_i| = 0 \text{ or } 1 \text{ for } i = 1,\dots,n. \tag{3.9}$$

Then T_i^n is equal to the quotient space $I_i^n / \sim_i$ for $i = 1,\dots,n$. Let l_0 be the line segment joining origin and the vector e_0 of $\mathbb{R}^n$. Consider the triangulation of l_0 with vertices

$$\{(l,\dots,l) : l \in \{0,a,b,1\}\}.$$

This induces a $\mathbb{Z}_3$-equivariant triangulation of the circle S^1_0. So we have a free $\mathbb{Z}_3^n$-action of T_i^n via the diagonal action.

Lemma 3.4. *The triangulation on I^n induces a triangulation on $I^n \cap I_i^n$ for each $i \in \{1,\dots,n\}$.*

Proof. The intersection $I^n \cap J_i^n$ is the convex hull of $\{\mathbf{0}, \mathbf{e}, \dots, \mathbf{e_{i-1}}, \mathbf{e}, \mathbf{e_{i+1}}, \dots, \mathbf{e_n}\}$ in $\mathbb{R}^n$. Then the lemma follows from the construction of triangulation of I^n. $\square$

Lemma 3.5. *Fix $k \in \{1, \ldots, n\}$. There is a $\mathbb{Z}_3^n$-equivariant triangulations of $\mathbb{T}_k^n$ with the same simplices of the triangulation $\mathcal{T}^n$ of $\mathbb{T}^n$.*

Proof. Let $K = \{\sigma \subseteq \{1, \ldots, k-1, k+1, \ldots, n\}\}$ and $\sigma^c = \{1, \ldots, k-1, k+1, \ldots, n\} - \sigma$. Let

$$e_\sigma = (\epsilon_1, \ldots, \epsilon_n) \text{ where } \epsilon_i = 1 \text{ if } i \in \sigma \text{ and } \epsilon_i = 0 \text{ if } i \notin \sigma. \tag{3.10}$$

For each $\sigma \in K$, we define

$$A_\sigma = \{(x_1, \ldots, x_n) \in I_k^n : x_i \geq 1 \text{ if } i \in \sigma \text{ and } x_i \leq 1 \text{ if } i \in \sigma^c\}. \tag{3.11}$$

Then

$$I_k^n = \bigcup_{\sigma \in K} A_\sigma \text{ and } I^n = \bigcup_{\sigma \in K} \{A_\sigma - e_\sigma\}. \tag{3.12}$$

Note that $A_\sigma - e_\sigma$ is the convex hull of some vertices of

$$\{\epsilon_1 e_1 + \cdots + \epsilon_n e_n : \epsilon_i \in \{0, 1\} \text{ for } i = 1, \ldots, n\}.$$

From the construction of the of triangulation $\mathcal{I}^n$ of I^n, we get that $A_\sigma - e_\sigma$ is a triangulated subcomplex of $\mathcal{I}^n$. Let $f_\sigma : A_\sigma - e_\sigma \to A_\sigma$ be a map defined by

$$(x_1, \ldots, x_n) \to (x_1, \ldots, x_n) + e_\sigma.$$

So the triangulation of $A_\sigma - e_\sigma$ induces a triangulation of A_σ via the map f_σ. Hence by equation 3.12, we get a triangulation $\mathcal{I}_k^n$ of I_k^n.

Now we denote the opposite vertices of I^n by the same. This gives the $\mathbb{Z}_3$-equivariant triangulation $\mathcal{T}^n$ of $\mathbb{T}^n$. We denote the image of a vertex under the map f_σ by the same. So we get a triangulation $\mathcal{I}_k^n$ where the points in the opposite facets are identified. Clearly, this gives a $\mathbb{Z}_3^n$-equivariant triangulation $\mathcal{T}_k^n$ of $\mathbb{T}_k^n$. Note that $\mathcal{T}_k^n$ is same as the triangulation $\mathcal{T}^n$. $\square$

Remark 3.6. *If σ is $\triangle^n$, then $I_\sigma = C_{\triangle^n}$. We consider the triangulation of $\pi^{-1}(I_\sigma)$ is $\mathcal{T}^n$.*

4. Some $\mathbb{Z}_3^n$-equivariant triangulation of complex projective spaces

In this section we construct some $\mathbb{Z}_3^n$-equivariant triangulation of $\mathbb{CP}^n$. Let $\triangle^n$ be an n-dimensional simplex with vertices $\{v_0, \ldots, v_n\}$. Let F_i be the facet not containing the vertex v_i of $\triangle^n$. Recall from Remark 2.2 that $\pi : \mathbb{CP}^n \cong M(\triangle^n, \xi) \to \triangle^n$ be the orbit map and I_σ is the cube in the cubical subdivision of $\triangle^n$ corresponding to the face σ of $\triangle^n$. Note that if σ is $\triangle^n$, then $\pi^{-1}(I_\sigma)$ is $\mathbb{T}^n$. To triangulate $\mathbb{CP}^n$, we construct $\mathbb{Z}_3^n$-equivariant triangulation of $\pi^{-1}(I_\sigma)$ such that $\pi^{-1}(I_\sigma)$ is a $\mathbb{Z}_3^n$-equivariant subcomplex of $\pi^{-1}(I_\tau)$ for any face τ of σ. We denote the cone on a topological space X by CX.

Lemma 4.1. *Let $B(k) = S_1^1 \times \cdots \times S_{k-1}^1 \times CS_0^1 \times S_{k+1}^1 \times \cdots \times S_n^1$ for $k = 1, \ldots, n$. There is a $\mathbb{Z}_3^n$-equivariant triangulation $\mathcal{B}(k)$ of $B(k)$ such that $\mathcal{T}_k^n$ is a subcomplex of $\mathcal{B}(k)$ for $k = 1, \ldots, n$.*

Proof. Let c_k be the center of $CS_0^1 \cong D^2 = \{z \in \mathbb{C} : |z| \leq 1\}$. Note that $\mathbb{T}_k^n$ is the boundary of $B(k)$ and $\mathbb{Z}_3^n$-action on $\mathbb{T}_k^n$ extend to an action on $B(k)$ for $k = 1, \ldots, n$. We consider the following natural isomorphism,

$$\iota_k : S_1^1 \times \cdots \times S_{k-1}^1 \times c_k \times S_{k+1}^1 \times \cdots \times S_n^1 \to S_1^1 \times \cdots \times S_{k-1}^1 \times 1 \times S_{k+1}^1 \times \cdots \times S_n^1 \subset \mathbb{T}^n.$$

This induces a $\mathbb{Z}_3^{n-1}$-equivariant triangulation on

$$S_1^1 \times \cdots \times S_{k-1}^1 \times c_k \times S_{k+1}^1 \times \cdots \times S_n^1.$$

We denote the image of

$$[0,a]\times\cdots\times[0,a]\times[\mathbf{0},\mathbf{a}]\times[0,a]\times\cdots\times[0,a]\to T_k^n$$

by the same where $\mathbf{0}=(0,\ldots,0)$ and $\mathbf{a}=(a,\ldots,a)$ in I^n. So

$$[0,a]\times\cdots\times[0,a]\times C[\mathbf{0},\mathbf{a}]\times[0,a]\times\cdots\times[0,a]$$

is a fundamental cell of $B(k)$. Let $C[\mathbf{0},\mathbf{a}]$ be the triangle with vertices $c_k,\mathbf{0}$ and $\mathbf{a}$. Note that the triangulation on the image (which we denote by the same) of

$$[0,a]\times\cdots\times[0,a]\times c_k\times[0,a]\times\cdots\times[0,a]\to B(k)$$

is induced from

$$[0,a]\times\cdots\times[0,a]\times 1\times[0,a]\times\cdots\times[0,a]\subset S_1^1\times\cdots\times S_{k-1}^1\times 1\times S_{k+1}^1\times\cdots\times S_n^1$$

via the map ι_k. Note that we have two embedding following,

$$\iota_c: S_1^1\times\cdots\times S_{k-1}^1\times c_k\times S_{k+1}^1\times\cdots\times S_n^1\to B(k) \tag{4.1}$$

$$\iota_b: S_1^1\times\cdots\times S_{k-1}^1\times S_0^1\times S_{k+1}^1\times\cdots\times S_n^1\to B(k). \tag{4.2}$$

These two maps induces $\mathbb{Z}_3^{n-1}$- and $\mathbb{Z}_3^n$-equivariant triangulation on the image of ι_c and ι_b respectively. So we have triangulation on

$$[0,a]\times\cdots\times[0,a]\times c_k\times[0,a]\times\cdots\times[0,a]\subset B(k)$$
$$\text{and } [0,a]\times\cdots\times[0,a]\times[\mathbf{0},\mathbf{a}]\times[0,a]\times\cdots\times[0,a]\subset B(k).$$

Observe that

$$D(a,\ldots,a,\mathbf{0},a,\ldots,a):=[0,a]\times\cdots\times[0,a]\times[c_k,\mathbf{0}]\times[0,a]\times\cdots\times[0,a]$$
$$\text{and } D(a,\ldots,a,\mathbf{a},a,\ldots,a):=[0,a]\times\cdots\times[0,a]\times[c_k,\mathbf{a}]\times[0,a]\times\cdots\times[0,a]$$

are cell complexes such that each cell is an n-dimensional prism, that is each cell homeomorphic to $\triangle^{n-1}\times[c_k,\mathbf{0}]$ and $\triangle^{n-1}\times[c_k,\mathbf{a}]$ respectively. So by Proposition 2.7, we can triangulate $D(a,\ldots,a,\mathbf{0},a,\ldots,a)$ without adding extra vertices. Note that

$$D(a,\ldots,a,\mathbf{a},a,\ldots,a)=gD(a,\ldots,a,\mathbf{0},a,\ldots,a)$$

for some $g\in\mathbb{Z}_3^n$. This induces a triangulation on $D(a,\ldots,a,\mathbf{a},a,\ldots,a)$ without adding extra vertices. Again applying Proposition 2.7, we can triangulate

$$E(k):=[0,a]\times\cdots\times[0,a]\times C[\mathbf{0},\mathbf{a}]\times[0,a]\times\cdots\times[0,a]\subset B(k)$$

without adding extra vertices. We have,

$$B(k)=\bigcup_{g\in\mathbb{Z}_3^n} gE(k).$$

So this induces a $\mathbb{Z}_3^n$-equivariant triangulation $\mathcal{B}(k)$ on $B(k)$. □

Remark 4.2. *Note that*

$$S_1^1\times\cdots\times S_{k-1}^1\times c_k\times S_{k+1}^1\times\cdots\times S_n^1$$

and

$$S_1^1\times\cdots\times S_{l-1}^1\times c_l\times S_{l+1}^1\times\cdots\times S_n^1$$

are isomorphic which is obtained by flipping the k-th and l-th coordinates. We denote the images of the vertices under this isomorphism by the same. Then the $\mathbb{Z}_3^n$-equivariant triangulation on $B(k)$ and $B(l)$ are same for $k,l\in\{1,\ldots,n\}$.

Remark 4.3. *Let σ be the face not containing the vertex v_0 of $\triangle^n$. We assign the $\mathbb{Z}_3^n$-equivariant triangulation $\mathcal{B}(k)$ to the set $\pi^{-1}(I_\sigma)$.*

Proof of the following lemma is similar to the previous lemma. So we omit the proof.

Lemma 4.4. *Let $C(k) = S^1_1 \times \cdots \times S^1_{k-1} \times CS^1_k \times S^1_{k+1} \times \cdots \times S^1_n$ for $k = 1, \ldots, n$. There is a $\mathbb{Z}^n_3$-equivariant triangulation $\mathcal{C}(k)$ of $C(k)$ such that $\mathcal{T}^n$ is a subcomplex of $\mathcal{C}(k)$ for $k = 1, \ldots, n$.*

Remark 4.5. *Let σ be the face not containing the vertex v_k of $\triangle^n$. We assign the $\mathbb{Z}^n_3$-equivariant triangulation $\mathcal{C}(k)$ to the set $\pi^{-1}(I_\sigma)$.*

Let $\{i_1, \ldots, i_s\}$ be a subset of $\{1, \ldots, k-1, k+1, \ldots, n\}$ with $i_1 < \cdots < i_s$. Let $B(k, i_1, \ldots, i_s)$ be the set

$$S^1_1 \times \cdots \times S^1_{i_1-1} \times CS^1_{i_1} \times S^1_{i_1+1} \times \cdots \times S^1_{k-1} \times CS^1_0 \times S^1_{k+1} \times \cdots \times S^1_{i_k-1} \times CS^1_{i_k} \times S^1_{i_k} \times \cdots \times S^1_n.$$

Lemma 4.6. *There is a $\mathbb{Z}^n_3$-equivariant triangulation $\mathcal{B}(k, i_1, \ldots, i_s)$ of $B(k, i_1, \ldots, i_s)$ such that the triangulation $\mathcal{B}(k, j_1, \ldots, j_r)$ of $B(k, j_1, \ldots, j_r)$ is a $\mathbb{Z}^n_3$-equivariant subcomplex of $\mathcal{B}(k, i_1, \ldots, i_s)$ for any subset $\{j_1, \ldots, j_r\}$ of $\{i_1, \ldots, i_k\}$.*

Proof. We prove the lemma when $r = 0$ and $s = 1$. The general case is similar to this proof. We may assume $i_1 < k$. By lemma 4.1 and 4.4 there is a $\mathbb{Z}^n_3$-equivariant triangulation of $B(k)$ and $\mathbb{Z}^{n-1}_3$-equivariant triangulation of

$$S^1_1 \times \cdots \times S^1_{i_1-1} \times CS^1_{i_1} \times S^1_{i_1+1} \times \cdots \times S^1_{k-1} \times 1 \times S^1_{k+1} \times \cdots \times S^1_n$$

respectively. Note that $\mathbb{Z}^n_3$-action on $B(k)$ extends naturally to an action on $B(k, i_1)$.

Let c_{i_1} be the center of $CS^1_{i_2} \cong D^2$. We consider a triangulation on

$$S^1_1 \times \cdots \times S^1_{i_1-1} \times c_{i_1} \times S^1_{i_1+1} \times \cdots \times S^1_{k-1} \times CS^1_0 \times S^1_{k+1} \times \cdots \times S^1_n$$

induced from the $\mathbb{Z}^{n-1}_3$-equivariant triangulation of

$$S^1_1 \times \cdots \times S^1_{i_1-1} \times 1 \times S^1_{i_1+1} \times \cdots \times S^1_{k-1} \times CS^1_0 \times S^1_{k+1} \times \cdots \times S^1_n \subset B(k)$$

via the natural isomorphism. We denote the image of

$$[0, a] \times \cdots \times [0, a] \times C[\mathbf{0}, \mathbf{a}] \times [0, a] \times \cdots \times [0, a] \to B(k)$$

by the same where $\mathbf{0} = (0, \ldots, 0)$ and $\mathbf{a} = (a, \ldots, a)$ in I^n. So

$$[0, a] \times \cdots \times [0, a] \times C[0, a] \times [0, a] \times \cdots \times [0, a] \times C[\mathbf{0}, \mathbf{a}] \times [0, a] \times \cdots \times [0, a]$$

is a fundamental cell of $B(k, i_1)$ where $C[0, a] \subset CS^1_{i_1}$. Let $C[0, a]$ be the triangle with vertices $c_{i_1}, 0$ and a. Note that we have two embedding following,

$$\iota_{ci_1} : S^1_1 \times \cdots \times S^1_{i_1-1} \times c_{i_1} \times S^1_{i_1+1} \times \cdots \times S^1_{k-1} \times CS^1_0 \times S^1_{k+1} \cdots \times S^1_n \to B(k, i_1), \tag{4.3}$$

$$\iota_{bi_1} : S^1_1 \times \cdots \times S^1_{i_1-1} \times S^1_{i_1} \times S^1_{i_1+1} \times \cdots \times S^1_{k-1} \times CS^1_0 \times S^1_{k+1} \times \cdots \times S^1_n \to B(k, i_1). \tag{4.4}$$

These two maps induces $\mathbb{Z}^{n-1}_3$- and $\mathbb{Z}^n_3$-equivariant triangulation on the image of ι_{ci_1} and ι_{bi_1} respectively. So we have triangulation on

$$[0, a] \times \cdots [0, a] \times c_{i_1} \times [0, a] \times [0, a] \times C[\mathbf{0}, \mathbf{a}] \times [0, a] \times \cdots \times [0, a]$$

$$\text{and } [0, a] \times \cdots [0, a] \times [0, a] \times [0, a] \times [0, a] \times C[\mathbf{0}, \mathbf{a}] \times [0, a] \times \cdots \times [0, a].$$

Let $D(a, \ldots, a, 0, a, \ldots, a, \mathbf{0}, a, \ldots, a)$ be

$$[0, a] \times \cdots [0, a] \times [c_{i_1}, 0] \times [0, a] \times \cdots \times [0, a] \times [c_k, \mathbf{0}] \times [0, a] \times \cdots \times [0, a]$$

and $D(a, \ldots, a, a, a, \ldots, a, \mathbf{a}, a, \ldots, a)$ be

$$[0, a] \times \cdots [0, a] \times [c_{i_1}, a] \times [0, a] \times \cdots \times [0, a] \times [c_k, \mathbf{0}] \times [0, a] \times \cdots \times [0, a].$$

So $D(a, \ldots, a, 0, a, \ldots, a, \mathbf{0}, a, \ldots, a)$ and $D(a, \ldots, a, a, a, \ldots, a, \mathbf{a}, a, \ldots, a)$ are cell complexes such that each cell is an $(n+1)$-dimensional prism, that is each cell is homeomorphic

to $\triangle^n \times [c_{i_1}, 0]$ and $\triangle^n \times [c_{i_1}, a]$ respectively. So by Proposition 2.7, we can triangulate $D(a, \ldots, a, 0, a, \ldots, a, \mathbf{0}, a, \ldots, a)$ without adding extra vertices. Note that

$$D(a, \ldots, a, a, a, \ldots, a, \mathbf{a}, a, \ldots, a) = gD(a, \ldots, a, 0, a \ldots, a, \mathbf{0}, a, \ldots, a)$$

for some $g \in \mathbb{Z}_3^n$. This induces a triangulation on $D(a, \ldots, a, \mathbf{a}, a, \ldots, a)$. Again applying Proposition 2.7, we can triangulate

$$E(k, i_1) := [0, a] \times \cdots [0, a] \times C[0, a] \times [0, a] \times \cdots \times [0, a] \times C[\mathbf{0}, \mathbf{a}] \times [0, a] \times \cdots \times [0, a]$$

without adding extra vertices. We have,

$$B(k, i_1) = \bigcup_{g \in \mathbb{Z}_3^n} gE(k, i_1).$$

So this induces a $\mathbb{Z}_3^n$-equivariant triangulation $\mathcal{B}(k, i_1)$ on $B(k, i_1)$ such that $B(k)$ is a $\mathbb{Z}_3^n$-equivariant subcomplex of $B(k, i_1)$. In general, we denote the triangulation of $B(k, i_1, \ldots, i_s)$ by $\mathcal{B}(k, i_1, \ldots, i_s)$. □

Remark 4.7. *Note that there is a natural inclusion $C(i_1) \to B(k, i_1)$. By careful choice of the vertices in the construction of $\mathcal{B}(k, i_1)$, we can show that $\mathcal{C}(i_1)$ is also a subcomplex of $\mathcal{B}(k, i_1)$.*

Remark 4.8. *By similar arguments as in Remark 4.2, we can consider the triangulation $\mathcal{B}(k, i_1, \ldots, i_s)$ and $B(l, i_1, \ldots, i_s)$ are same for any $k, l \in \{1, \ldots, n\} - \{i_1, \ldots, i_s\}$.*

Remark 4.9. *Let $\{i_1, \ldots, i_s\}$ be a subset of $\{1, \ldots, k-1, k+1, \ldots, n\}$. Let σ be the face not containing the vertices $\{v_0, v_{i_1}, \ldots, v_{i_s}\}$ of $\triangle^n$. We assign the triangulation $\mathcal{B}(k, i_1, \ldots, i_s)$ to the subset $\pi^{-1}(I_\sigma)$ of $\mathbb{CP}^n$.*

Let $\{i_1, \ldots, i_k\}$ be a subset of $\{1, \ldots, n\}$. Let

$$C(i_1, \ldots, i_k) = S_1^1 \times \cdots \times S_{i_1 - 1}^1 \times CS_{i_1}^1 \times S_{i_1+1}^1 \times \cdots \times S_{i_k - 1}^1 \times CS_{i_k}^1 \times S_{i_k}^1 \times \cdots \times S_n^1.$$

Proof of the following lemma is similar to the Lemma 4.6. So we omit the proof.

Lemma 4.10. *There is a $\mathbb{Z}_3^n$-equivariant triangulation $\mathcal{C}(i_1, \ldots, i_k)$ of $C(i_1, \ldots, i_k)$ such that the triangulation $\mathcal{C}(j_1, \ldots, j_l)$ of $C(j_1, \ldots, j_l)$ is a $\mathbb{Z}_3^n$-equivariant subcomplex of the triangulation $\mathcal{C}(i_1, \ldots, i_k)$ for any subset $\{j_1, \ldots, j_l\}$ of $\{i_1, \ldots, i_k\}$.*

Remark 4.11. *Let $\{i_1, \ldots, i_k\}$ be a subset of $\{1, \ldots, n\}$. Let σ be the face not containing the vertices $\{v_{i_1}, \ldots, v_{i_k}\}$ of $\triangle^n$. We assign the triangulation $\mathcal{C}(i_1, \ldots, i_s)$ to the subset $\pi^{-1}(I_\sigma)$ of $\mathbb{CP}^n$.*

Theorem 4.12. *There is a $\mathbb{Z}_3^n$-equivariant triangulation of $\mathbb{CP}^n$ with $\frac{4^{n+1}-1}{3}$ vertices for all n.*

Proof. Construction of $\mathbb{Z}_3^n$-equivariant triangulation of $\mathbb{CP}^n$ follows from the Remarks 3.6, 4.3, 4.5, 4.9, 4.11 and the Equation 2.4. Observe that vertices of the triangulation of $\pi^{-1}(I_\sigma)$ belong to $\pi^{-1}(C_\tau)$ where σ is a face of τ. If σ is a k-dimensional face, then $\pi^{-1}(C_\sigma)$ is a k-dimensional torus. By our construction $\pi^{-1}(C_\sigma)$ contains 3^k many vertices of the triangulation of $\mathbb{CP}^n$. □

Example 4.13. We construct some $\mathbb{Z}_3^3$-equivariant triangulation of $\mathbb{CP}^3$ following theorem 4.12. Let $\triangle^3$ be the 3-simplex with vertices $\{v_0, \ldots, v_3\}$. Let F_i be the facet not containing the vertex v_i for $i = 0, \ldots, 3$. Let $\pi : \mathbb{CP}^3 \to \triangle^3$ be the orbit map. The characteristic vectors are given by,

$$F_0 \to e_0 := (1, 1, 1), \; F_1 \to e_1 = (1, 0, 0), \; F_2 \to e_2 = (0, 1, 0), \text{ and } F_3 \to e_3 = (0, 0, 1).$$

Let S_i^1 be the circle subgroup of $\mathbb{T}^3$ determined by the vector e_i in $\mathbb{R}^3$ for $i = 0, \ldots, 3$. For each face σ of $\triangle^3$ we find $\pi^{-1}(I_\sigma)$ in the following.

(1) If $\sigma = \{v_0, \ldots, v_3\}$, then
$$\pi^{-1}(I_\sigma) = S^1_1 \times S^1_2 \times S^1_3.$$
(2) If $\sigma = \{v_0, v_2, v_3\}$, then
$$\pi^{-1}(I_\sigma) = CS^1_1 \times S^1_2 \times S^1_3.$$
(3) If $\sigma = \{v_0, v_1, v_3\}$, then
$$\pi^{-1}(I_\sigma) = S^1_1 \times CS^1_2 \times S^1_3.$$
(4) If $\sigma = \{v_0, v_1, v_2\}$, then
$$\pi^{-1}(I_\sigma) = S^1_1 \times S^1_2 \times CS^1_3.$$
(5) If $\sigma = \{v_1, v_2, v_3\}$, then
$$\pi^{-1}(I_\sigma) = CS^1_0 \times S^1_2 \times S^1_3 \cong S^1_1 \times CS^1_0 \times S^1_3 \cong S^1_1 \times S^1_2 \times CS^1_0.$$
(6) If $\sigma = \{v_0, v_1\}$, then
$$\pi^{-1}(I_\sigma) = S^1_1 \times CS^1_2 \times CS^1_3.$$
(7) If $\sigma = \{v_0, v_2\}$, then
$$\pi^{-1}(I_\sigma) = CS^1_1 \times S^1_2 \times CS^1_3.$$
(8) If $\sigma = \{v_0, v_3\}$, then
$$\pi^{-1}(I_\sigma) = CS^1_1 \times CS^1_2 \times S^1_3.$$
(9) If $\sigma = \{v_1, v_2\}$, then
$$\pi^{-1}(I_\sigma) = S^1_1 \times CS^1_0 \times CS^1_3 \cong CS^1_e \times S^1_2 \times CS^1_3.$$
(10) If $\sigma = \{v_1, v_3\}$, then
$$\pi^{-1}(I_\sigma) = S^1_1 \times CS^1_2 \times CS^1_0 \cong CS^1_0 \times CS^1_2 \times S^1_3.$$
(11) If $\sigma = \{v_2, v_3\}$, then
$$\pi^{-1}(I_\sigma) = CS^1_1 \times CS^1_0 \times S^1_3 \cong CS^1_1 \times S^1_2 \times CS^1_0.$$
(12) If $\sigma = \{v_0\}$, then
$$\pi^{-1}(I_\sigma) = CS^1_1 \times CS^1_2 \times CS^1_3.$$
(13) If $\sigma = \{v_1\}$, then
$$\pi^{-1}(I_\sigma) = CS^1_0 \times CS^1_2 \times CS^1_3.$$
(14) If $\sigma = \{v_2\}$, then
$$\pi^{-1}(I_\sigma) = CS^1_1 \times CS^1_0 \times CS^1_3.$$
(15) If $\sigma = \{v_3\}$, then
$$\pi^{-1}(I_\sigma) = CS^1_1 \times CS^1_2 \times CS^1_0.$$

Consider the triangulation $\mathcal{T}^3$ for $\mathbb{T}^3$ as constructed in Lemma 3.2. Following Lemma 3.2, 4.1, 4.4, 4.6 and 4.10, we can construct $\mathbb{Z}_3^3$-equivariant triangulation of $\pi^{-1}(I_\sigma)$ for each face σ of $\triangle^3$ such that the $\mathbb{Z}_3^n$-equivariant triangulation of $\pi^{-1}(I_\sigma)$ is a equivariant subcomplex of the $\mathbb{Z}_3^n$-equivariant triangulation of $\pi^{-1}(I_\tau)$ if τ is a face of σ.

Remark 4.14. *Let Q be an n-dimensional simple polytope with facets $\{F_1, \ldots, F_m\}$. Let*

$$\xi : \{F_1, \ldots, F_m\} \to \{e_0, e_1, \ldots, e_n\}$$

be a standard characteristic function of Q. Let $M(Q, \xi)$ be the corresponding toric manifold and $\pi : M(Q, xi) \to Q$ be the orbit map.

We consider the $\mathbb{Z}_3^n$-equivariant triangulation of $\pi^{-1}(I_Q)$ is $\mathcal{T}^n$. Let σ be a face of Q of codimension l. So $\sigma = F_{i_1} \cap \cdots \cap F_{i_l}$ for a unique collection of facets $\{F_{i_1}, \ldots, F_{i_l}\}$. Let $\xi(F_{i_s}) = e_{j_s}$ for $s = 1, \ldots, l$. If $0 \notin \{j_1, \ldots, j_l\}$, then we assign the $\mathbb{Z}_3^n$-equivariant triangulation $\mathcal{C}(j_1, \ldots, j_l)$ of Lemma 4.10 to the subset $\pi^{-1}(I_\sigma)$. Let $0 \in \{j_1, \ldots, j_l\}$. So there is an $k \in \{1, \ldots, n\} - \{j_1, \ldots, j_l\}$. We may assume $j_1 = 0$. Then we assign the $\mathbb{Z}_3^n$-equivariant triangulation $\mathcal{B}(k, j_2, \ldots, j_l)$ of Lemma 4.6 to the subset $\pi^{-1}(I_\sigma)$. By a suitable choice of vertices we can consider the $\mathbb{Z}_3^n$-equivariant triangulation of $\pi^{-1}(I_\sigma)$ is a equivariant subcomplex of the $\mathbb{Z}_3^n$-equivariant triangulation of $\pi^{-1}(I_\tau)$ if τ is a face of σ. Then by Equation 2.4, we can construct a $\mathbb{Z}_3^n$-equivariant triangulation of $M(Q, \xi)$.

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